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A Polynomial Approach to Topological Analysis. III

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1. INTRODUCTION

In this paper, employing properties of complex polynomials, we give a construction of the topological index (winding number). In previous papers, [1] and [2], a theory of complex functions based upon polynomial methods was developed for continuously differentiable functions. This construction enables absorbing into the polynomial approach the methods of G. T. Whyburn [5], thus allowing the handling of the general case where no condition of continuity is placed on the derivative.

Let K be the (open) complex plane. For $\delta > 0$, $z_0 \in K$, set $U_{z_0}(\delta) = \{z \in K;\ |z - z_0| < \delta\}$, $B_{z_0}(\delta) = \{z \in K;\ |z - z_0| = \delta\}$, $U(\delta) = U_0(\delta)$, $U = U(1)$, $B(\delta) = B_0(\delta)$, and $B = B(1)$.

Let H be a compact subset of K. Then $C(H)$ denotes the space of continuous functions on H into K with the maximum norm, and $C'(H)$ denotes the family of all elements of $C(H)$ which never vanish on H. For $f \in C(B)$, set $\|f\| = \max\{|f(z)|; z \in B\}$, $\|f\|_{\min} = \min\{|f(z)|; z \in B\}$.

Let T be the family of all functions of $C(B)$ of the form $\sum_{-n}^{n} a_k z^k$, $n = 0, 1, \ldots$. Finally, set $T' = T \cap C'(B)$.

2. THE INDEX

Let T_0 be the family of all real-valued elements of T. Clearly, T_0 is a subalgebra of $C(B)$ containing the constant function 1. Let $z_1, z_2 \in B$, $z_1 \neq z_2$, and set, for $z \in B$, $P(z) = |z - z_1|^2 = (z-z_1)(\bar{z} - \bar{z}_1) = -z_1 z^{-1} + 2 - \bar{z}_1 z$. Then $P \in T_0$ and $P(z_2) = |z_2 - z_1|^2 > 0$. Thus, T_0 separates points of B. Hence, from the Stone–Weierstrass Theorem, the closure $\overline{T}_0$ of T_0 is the space of all real-valued elements of $C(B)$. Whence $\overline{T} = C(B)$.

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For $f(z) = \sum_{-n}^{n} a_k z^k \in T$, set $L_0(f) = a_0$. Then (cf. [2, Theorem 3.1]) L_0 is a bounded linear functional on T, and hence L_0 can be extended uniquely to a bounded linear functional L on the closure $\bar{T} = C(B)$ of T. For $f(z) = \sum_{-n}^{n} a_k z^k \in T'$, set $\tilde{n}(f) = L(zf'(z)/f(z))$.

THEOREM 1. *Let P be a polynomial such that $P(z) \neq 0$ throughout B. Then $\tilde{n}(P)$ is the number of zeros of P in U.*

Proof. We may assume that $P(z)$ is not a constant. Let $P(z) = \alpha(z - z_1) \cdots (z - z_m)$, so that $P'(z)/P(z) = \sum_1^m (z - z_k)^{-1}$ for $z \in B$.

Let y be a zero of P. If $y \in U$, then for $z \in B$, $z(z - y)^{-1} = (1 - yz^{-1})^{-1} = 1 + \sum_1^\infty y^n z^{-n}$ and, hence, $L(z(z - y)^{-1}) = 1$. If $y \in K - \bar{U}$, then for $z \in B$, $z(z - y)^{-1} = -zy^{-1}[1 - (z/y)]^{-1} = -zy^{-1} \sum_0^\infty z^n y^{-n} = \sum_1^\infty z^n y^{-n}$ and, hence, $L(z(z - y)^{-1}) = 0$. Thus $\tilde{n}(P) = L(zP'(z)/P(z)) = \sum_1^m L(z(z - z_k)^{-1})$ is the number of k's with $z_k \in U$; consequently, $\tilde{n}(P)$ is the number of zeros of P in U.

THEOREM 2. *For $f, g \in T'$:*

1. *$\tilde{n}(f)$ is an integer.*
2. *$\tilde{n}(fg) = \tilde{n}(f) + \tilde{n}(g)$ and $\tilde{n}(f/g) = \tilde{n}(f) - \tilde{n}(g)$.*
3. *$|f(z) - g(z)| < |f(z)|$ for all $z \in B$ implies $\tilde{n}(f) = \tilde{n}(g)$.*

Proof. Set $h = fg$. Then $h'/h = (f'/f) + (g'/g)$ and $\tilde{n}(h) = L(zf'(z)/f(z)) + L(zg'(z)/g(z)) = \tilde{n}(f) + \tilde{n}(g)$. Similarly, $\tilde{n}(f/g) = \tilde{n}(f) - \tilde{n}(g)$.

For some integer p, there exists a polynomial P such that $P(z) = z^p f(z)$. Then $\tilde{n}(P) = \tilde{n}(z^p) + \tilde{n}(f) = p\tilde{n}(z) + \tilde{n}(f) = p + \tilde{n}(f)$. From Theorem 1, $\tilde{n}(f)$ is the integer $\tilde{n}(P) - p$.

For $t \in [0, 1]$, $z \in B$, set $h_t(z) = h(t, z) = (1 - t) f(z) + tg(z)$. Under the hypothesis of 3, for all such t and z,

$$|h(t, z)| \leqslant |f(z)| - t\,|f(z) - g(z)| \geqslant |f(z)| - |f(z) - g(z)| > 0.$$

For $t \in [0, 1]$, $z \in B$, set $\Phi_t(z) = \Phi(t, z) = zh_t'(z)/h_t(z)$. Then $\Phi_0(z) = zf'(z)/f(z)$ and $\Phi_1(z) = zg'(z)/g(z)$ for $z \in B$. For $t \in [0, 1]$, set $w(t) = \tilde{n}(h_t) = L(z\Phi_t(z))$. Since L is a bounded linear operator, w is continuous. Since $w(t)$ is an integer, w is a constant and $\tilde{n}(f) = w(0) = w(1) = \tilde{n}(g)$.

THEOREM 3. *Let $f \in C'(B)$. Then there is a unique integer $n(f)$ (the topological index of f) such that $P \in T'$ and $|f(z) - P(z)| < 4^{-1}|f(z)|$ for all $z \in B$ imply $n(f) = \tilde{n}(P)$.*

Proof. Let $\sigma = \|f\|_{\min} > 0$. By the Stone–Weierstrass theorem, there exists $Q \in T$ such that $\|Q - f\| < \sigma/4$. Set $n(f) = \tilde{n}(Q)$. For all $z \in B$,

$|Q(z) - f(z)| < 4^{-1}\sigma \leqslant 4^{-1}|f(z)|$. Now $Q \in T'$, since, if for some $z \in B$, $Q(z) = 0$, we would have $|f(z)| = |Q(z) - f(z)| < |f(z)|$.

Now for $z \in B$, $4^{-1}|f(z)| > |f(z)| - |Q(z)|$ and so, $|Q(z)| > (3/4)|f(z)|$; thus, if $P \in T'$ and $|f(z) - P(z)| < 4^{-1}|f(z)|$ throughout B, then

$$\begin{aligned} |P(z) - Q(z)| &\leqslant |P(z) - f(z)| + |f(z) - Q(z)| \\ &< 4^{-1}|f(z)| + 4^{-1}|f(z)| < (3/4)|f(z)| < Q(z). \end{aligned}$$

From Theorem 2, $\tilde{n}(P) = \tilde{n}(Q) = n(f)$.

THEOREM 4. *For $f, g \in C'(B)$:*

1. *$n(fg) = n(f) + n(g)$ and $n(f/g) = n(f) - n(g)$.*
2. *$|f(z) - g(z)| < |f(z)|$ for all $z \in B$ implies $n(f) = n(g)$.*

Proof. Let $P, Q \in T'$ be such that for all $z \in B$,

$$|P(z) - f(z)| < 9^{-1}|f(z)| \qquad \text{and} \qquad |Q(z) - g(z)| < 9^{-1}|g(z)|.$$

Then for all $z \in B$, $9^{-1}|g(z)| > |Q(z)| - |g(z)|$ and $|Q(z)| < (10/9)|g(z)|$; thus

$$\begin{aligned} |P(z)Q(z) - f(z)g(z)| &\leqslant |P(z)Q(z) - Q(z)f(z)| + |Q(z)f(z) - f(z)g(z)| \\ &\leqslant (10/9)|g(z)| \cdot 9^{-1}|f(z)| + |f(z)| \cdot 9^{-1}|g(z)| \\ &= (19/81)|f(z)| \cdot |g(z)| < 4^{-1}|f(z)g(z)|. \end{aligned}$$

Hence, from Theorem 3, $n(fg) = n(PQ)$, $n(f) = n(P)$, $n(g) = n(Q)$. From Theorem 2, $n(PQ) = n(P) + n(Q)$. Thus,

$$n(fg) = n(PQ) = n(P) + n(Q) = n(f) + n(g).$$

Also $n(f) = n(g(f/g)) = n(g) + n(f/g)$ and, so, $n(f/g) = n(f) - n(g)$.

Assume now that for all $z \in B$, $|f(z)| - |f(z) - g(z)| > 0$. Set $\sigma = 3^{-1}\| |f| - |f - g| \|_{\min} > 0$. Then $|f(z) - g(z)| \leqslant |f(z)| - 3\sigma$ for all $z \in B$. Choose $P, Q \in T'$ such that $\|P - f\| < \min(\sigma, 4^{-1}\|f\|_{\min})$, $\|Q - g\| < \min(\sigma, 4^{-1}\|g\|_{\min})$. Then from Theorem 3, $n(P) = n(f)$ and $n(Q) = n(g)$. Thus, for every $z \in B$, $\sigma > \|f - P\| \geqslant |f(z)| - |P(z)|$ and $|f(z)| - \sigma < |P(z)|$; hence

$$\begin{aligned} |P(z) - Q(z)| &\leqslant |P(z) - f(z)| + |f(z) - g(z)| + |g(z) - Q(z)| \\ &\leqslant \|P - f\| + [|f(z)| - 3\sigma] + \|Q - g\| \\ &< \sigma + [|f(z)| - 3\sigma] + \sigma = |f(z)| - \sigma < |P(z)|. \end{aligned}$$

From Theorem 2, $n(P) = n(Q)$, and thus $n(f) = n(P) = n(Q) = n(g)$.

Let $f, g \in C'(B)$. We recall that f is homotopic to g in $C'(B)$ $(f \simeq g)$ if there exists a continuous function h on $[0, 1] \times B$ into $K - \{0\}$ such that for all $z \in B$ we have $h_0(z) = h(0, z) = f(z)$ and $h_1(z) = h(1, z) = g(z)$.

THEOREM 5. *Let $f, g \in C'(B)$ and let $f \simeq g$. Then $n(f) = n(g)$.*

Proof. Let h be a continuous function on $[0, 1] \times B$ into $K - \{0\}$ such that $h_0 = f$ and $h_1 = g$. Then $\alpha = \min\{|h(t, z)|; 0 \leqslant t \leqslant 1, z \in B\} > 0$. Let $\delta > 0$ be such that $\| h_s - h_t \| < \alpha$ whenever $s \in [0, 1]$, $t \in [0, 1]$ and $| s - t | < \delta$. For such s and t, since $| h_s(z) - h_t(z)| < \alpha \leqslant | h_t(z)|$ for all $z \in B$, we have, from Theorem 4, $n(h_s) = n(h_t)$. Thus, by suitably subdividing $[0, 1]$, we conclude that $n(f) = n(h_0) = n(h_1) = n(g)$.

THEOREM 6. *If $f \in C'(\overline{U})$, then $n(f) = 0$.*

Proof. For $t \in [0, 1]$, $z \in B$, set $h(t, z) = f(tz)$. Then $h_1(z) = h(1, z) = f(z)$ and $h_0(z) = h(0, z) = f(0)$ for all $z \in B$. Hence, from Theorem 5, $n(f) = n(h_1) = n(h_0) = L(zh_0'(z)/h(z)) = L(0) = 0$.

3. CONTINUITY OF THE DERIVATIVE

THEOREM 7. *Let $f \in C(\overline{U})$ be differentiable at $z_0 \in U$, and let $f'(z_0) \neq 0$. Then there exists δ, $0 < \delta < 1 - | z_0 |$, such that:*

1. *$z \in \overline{U}_{z_0}(\delta)$, $z \neq z_0$ imply $f(z) \neq f(z_0)$.*
2. *$H = f(\overline{U}_{z_0}(\delta))$ contains a disc about $f(z_0)$.*

Proof. Let $0 < \eta < 2^{-1} |f'(z_0)|$. Then there exists δ, $0 < \delta < 1 - | z_0 |$, such that for $z \in V = \overline{U}_{z_0}(\delta) - \{z_0\}$, we have

$$|[f(z) - f(z_0)] \cdot (z - z_0)^{-1} - f'(z_0)| < \eta.$$

Whence, for $z \in V$,

$$\begin{aligned} \eta \mid z - z_0 \mid &> | f(z) - f(z_0) - f'(z_0)(z - z_0)| \\ &\geqslant | f'(z_0)| \cdot | z - z_0 | - | f(z) - f(z_0)| \\ &> 2\eta \mid z - z_0 \mid - | f(z) - f(z_0)|, \end{aligned}$$

so that $| f(z) - f(z_0)| > \eta \mid z - z_0 \mid > 0$; thus $f(z) \neq f(z_0)$.

Let $\alpha \in U(\eta\delta)$ and set $h(z) = f(z_0 + z\delta) - f(z_0) - \alpha$ for $z \in \overline{U}$. Then, for $z \in B$,

$$\begin{aligned} | h(z) - f'(z_0) z\delta | &= | f(z_0 + z\delta) - f(z_0) - \alpha - f'(z_0) z\delta | \\ &\leqslant | f(z_0 + z\delta) - f(z_0) - f'(z_0) z\delta | + | \alpha | \\ &\leqslant \eta \mid z\delta \mid + | \alpha | \leqslant \eta \mid z\delta \mid + \eta\delta = 2\eta\delta < | f'(z_0) z\delta |. \end{aligned}$$

Hence, from Theorem 4, $n(h) = n(f'(z_0)\,\delta z) = n(f'(z_0)\,\delta) + n(z) = 1$. From Theorem 6, $h \notin C'(U)$, and thus there exists an $x \in U$ such that $h(x) = 0$. Whence $f(z_0 + x\delta) - f(z_0) - \alpha = 0$ and $\alpha + f(z_0) = f(z_0 + x\delta) \in H$. Thus $U_{f(z_0)}(\eta\delta) \subseteq H$.

THEOREM 8 (see Whyburn [5]). *Let $f \in C(\overline{U})$ and let $f'(z)$ exist and be zero throughout a set $V \subseteq U$. Then $f(V)$ has zero planar measure.*

THEOREM 9. *Let D be a countable subset of U and let $f \in C(\overline{U})$ be differentiable throughout $U - D$. Then for every $z \in U$, $|f(z)| \leqslant M = \|f\|$.*

Proof. Assume the contrary. Then for some $z_0 \in U - D$, $|f(z_0)| > M$. Set $\alpha = f(z_0)$. Let $0 < c < 2^{-1}(|\alpha| - M)$, $c \neq f'(z_0)$, and set $f_0(z) = f(z) + cz$ for every $z \in \overline{U}$. Then $f_0'(z_0) \neq 0$, $M_0 = \|f_0\| \leqslant M + c$, and $|f_0(z_0)| = |\alpha + cz_0| > |\alpha| - c > M + c \geqslant M_0$. Set $\alpha_0 = f_0(z_0)$ and let $0 < \delta < |\alpha_0| - M_0$. Then $S_0 = f^{-1}(U_{\alpha_0}(\delta) \cap f(\overline{U}))$ is a neighborhood of z_0. By Theorem 7, $f(S_0)$ contains a (nonempty) open set W.

Set $V = \{z \in U - D; f_0'(z) = 0\}$. By Theorem 8, $f_0(V)$ has zero planar measure. Since $f_0(D)$ is countable, $A = f_0(V) \cup f_0(D)$ has zero planar measure and, thus, $W - A$ is not empty. Let $\beta \in W - A$. Then, for every $x \in H = f_0^{-1}(\beta)$, $f_0'(x) \neq 0$. By Theorem 7, the points of H are isolated. Since H is closed, it is finite. Let $z_1, \ldots, z_k$ be the (distinct) points of H, set

$$\theta(z) = [f_0(z) - \beta]/[(z - z_1) \cdots (z - z_k)] \qquad \text{for} \quad z \in \overline{U} - H,$$

and

$$\theta(z_i) = f_0'(z_i)/[(z - z_1) \cdots (z - z_{i-1})(z - z_{i+1}) \cdots (z - z_k)] \qquad \text{for} \quad i = 1, \ldots, k.$$

For such an i, $f_0'(z_i) \neq 0$ and, thus, $\theta \in C'(\overline{U})$. From Theorems 1, 4, and 6,

$$n(f_0(z) - \beta) = n(\theta(z)(z - z_1) \cdots (z - z_k)) = n(\theta) + \sum_{i=1}^{k} n(z - z_i) = k > 0.$$

For $t \in [0, 1]$, $z \in B$, set $h(t, z) = (1 - t) f_0(z) - \beta$. Then

$$|\alpha_0| - |\beta| \leqslant |\alpha_0 - \beta| < \delta < |\alpha_0| - M_0 \qquad \text{and} \qquad |\beta| - M_0 > 0;$$

thus $|h(t, z)| \geqslant |\beta| - |f_0(z)| \geqslant |\beta| - M_0 > 0$ for $t \in [0, 1]$, $z \in B$. But then, from Theorem 5, $n(f_0(z) - \beta) = n(h_0) = n(h_1) = n(\beta) = 0$.

THEOREM 10. *Let $\delta > 0$, set $H = \overline{U}(1 + \delta)$, and let $f \in C(H)$ be differentiable throughout $U(1 + \delta)$. Then f' is continuous at 0. Thus differentiability of a function in an open subset of K implies continuity of the derivative there.*

Proof. Let $\eta > 0$. Then there exists δ_0, $0 < \delta_0 < \delta$, such that $s, t \in H$, $|s - t| < \delta_0$ imply $|f(s) - f(t)| < \eta/2$. Let $x \in U(\delta_0)$. For every $z \in \overline{U}$, $z \neq 0$, set $h_x(z) = [f(x + z) - f(x)]z^{-1} - [f(z) - f(0)]z^{-1}$. Let $h_x(0) = f'(x) - f'(0)$. For each $z \in B$, $|(x + z) - z| < \delta_0$ and, hence, $|f(x + z) - f(x)| < \eta/2$; thus, by Theorem 9,

$$\begin{aligned} |f'(x) - f'(0)| &= |h_x(0)| \leqslant \| h_x \| \\ &= \max\{|f(x + z) - f(x) - f(z) + f(0)|; z \in B\} \\ &\leqslant \max\{|f(x + z) - f(z)|; z \in B\} + |f(x) - f(0)| \\ &< (\eta/2) + (\eta/2) = \eta. \end{aligned}$$

Similar arguments were used by Porcelli and Connell [3], and Read [4] to obtain power series expansions, etc. Two further developments employing the Stone–Weierstrass theorem were given by the present author, [1] and [2].

4. Further Development of the Theory

In this section we study the fundamental homotopy group of $K - \{0\}$ and the relation between the local and the global degrees of a mapping.

THEOREM 11. *Let $f, g \in C'(B)$. Then $f \simeq z^q$, where $q = n(f)$. Moreover, $n(f) = n(g)$ implies $f \simeq g$.*

Proof. By Theorem 3, there exists an $f_0 \in T'$ such that $|f_0(z) - f(z)| < 4^{-1} |f(z)|$ for all $z \in B$, and $n(f_0) = n(f)$. Setting $h(t, x) = tf_0(z) + (1 - t) f(z)$ for $t \in [0, 1]$, $z \in B$, we have that $h(t, x) \neq 0$ for all $t \in [0, 1]$, $x \in B$, and, hence, $f_0 \simeq f$.

For some integer p, there exists a polynomial P such that $P(z) = z^p f_0(z)$ for $z \in B$. There exist $A(\neq 0)$, $z_1, \dots, z_k, z_{k+1}, \dots, z_{k_0}$ such that $P(z) = A(z - z_1) \cdots (z - z_{k_0})$, and such that $z_1, \dots, z_k \in U$ and $z_{k+1}, \dots, z_{k_0} \in K - \overline{U}$. For $t \in [0, 1]$, $z \in B$, set

$$h_t(z) = h(t, z) = A(z - tz_1) \cdots (z - tz_k) \cdot (tz - z_{k+1}) \cdots (tz - z_{k_0}) z^{-p}.$$

Then for $z \in B, h_1(z) = f_0(z)$ and $h_0(z) = \alpha z^{k-p}$, where $\alpha = A(-z_{k+1}) \cdots (-z_{k_0})$.

Now, for $z \in B$, $t \in [0, 1]$, $x \in U$, $y \in K - \overline{U}$, we have

$$| z - tx | \geqslant 1 - | x | > 0,$$

and

$$| tz - y | \geqslant | y | - 1 > 0.$$

Thus, $h(t, z) \neq 0$ for all $t \in [0, 1]$, $z \in B$. Hence, $f \simeq f_0 = h_1 \simeq h_0 = \alpha z^q \simeq z^q$, where $q = k - p$; from Theorem 5, $n(f) = n(\alpha z^q) = q$. Similarly, $g \simeq z^{n(g)}$, and hence, if $n(f) = n(g)$, we have $f \simeq z^{n(f)} \simeq g$.

By Theorem 4, n is a homomorphism of the multiplicative group $C'(B)$ onto the additive group Z of integers. By Theorems 5 and 11, for $f, g \in C'(B)$, $n(f) = n(g)$ if and only if $f \simeq g$. Thus the family $\overline{M}$ of homotopy classes of $C'(B)$ is isomorphic to Z.

This result is analogous to the theorem stating that the fundamental homotopy group $\overline{G}$ of $K - \{0\}$ is isomorphic to Z. The latter follows readily from the first if one shows that the operation of multiplying homotopy classes of $\overline{G}$ is independent of whether it is derived from pointwise multiplication of functions or from juxtapositioning of functions. The last statement follows from the fact that, by the first result, all homotopy classes of $\overline{G}$ are determined by functions of the form $e^{2pi\theta}$, $p \in Z$, $0 \leqslant \theta \leqslant 1$.

An $f \in C(\overline{U})$ will be called admissible if $f(z) \neq 0$ throughout B and if all zeros of f in U are isolated.

THEOREM 12. *Let $f \in C(\overline{U})$, and let $z_0 \in U$ be an isolated zero of f. Then there exists a unique integer $p = \mu(f, z_0)$ such that if $0 < \rho \leqslant 1 - |z_0|$ and if $z \in U_{z_0}(\rho) - \{z_0\}$ implies $f(z) \neq 0$, then $p = n(f(z_0 + \rho z))$.*

Proof. Let ρ be as in the theorem, and let σ satisfy, too, the same conditions. For $t \in [0, 1]$, $z \in B$, set $h_t(z) = h(t, z) = f(z_0 + [(1 - t)\rho + t\sigma] z)$. Then $h(t, z) \neq 0$ whenever $t \in [0, 1]$ and $z \in B$. Therefore, $f(z_0 + \rho z) = h_0 \simeq h_1 = f(z_0 + \sigma z)$, and from Theorem 5, $n(h_\sigma) = n(h_\rho)$.

$\mu(f, z_0)$ is the (local) degree of f at z_0.

For an admissible $f \in C(\overline{U})$, we set $\mu(f) = \sum_{f(z_0)=0} \mu(f, z_0)$. The integer $\mu(f)$ is the (glocal) degree of f in U.

THEOREM 13. *Let $f (\in C(\overline{U}))$ be admissible. Then $\mu(f) = n(f)$.*

Proof. Let z_0 be a zero of f in U and set $p = \mu(f, z_0)$. We shall modify f in a neighborhood S of z_0 in such a way that on some subneighborhood S_0 of S, f will take the form $A(z - z_0)^p$ if $p \geqslant 0$, and the form $A(\bar{z} - \bar{z}_0)^{-p}$ if $p < 0$, where $A \neq 0$.

For some δ, $0 < \delta \leqslant 1 - |z_0|$, $z \in U_{z_0}(\delta) - \{z_0\}$ implies $f(z) \neq 0$. Now $p = n(f(z_0 + \delta z))$, and by Theorem 11 there exists a continuous function Φ on $[0,1] \times B$ into $K - \{0\}$ such that $\Phi(1, z) = f(z_0 + \delta z)$ and $\Phi(0, z) = z^p$, for $z \in B$. Set $V = U_{z_0}(\delta) - \overline{U}_{z_0}(\delta/2)$, and

$$\theta(z) = \Phi(2\,|z - z_0|\,\delta^{-1} - 1, (z - z_0)\,|z - z_0|^{-1}) \qquad \text{for} \quad z \in \overline{V}.$$

Then, for $z \in B_{z_0}(\delta)$,

$$\theta(z) = \Phi(1, \delta^{-1}(z - z_0)) = f(z),$$

and for $z \in B_{z_0}(\delta/2)$,

$$\theta(z) = \Phi(0, 2(z - z_0)\,\delta^{-1}) = [2\delta^{-1}(z - z_0)]^p$$
$$= 2^p\delta^{-p}(z - z_0)^p = 2^{-p}\delta^p(\bar{z} - \bar{z}_0)^{-p}.$$

For $z \in \bar{U}$, we define $f_0(z)$, a modification of $f(z)$, to be:

$$\begin{array}{llll} f(z), & \text{if } z \notin U_{z_0}(\delta); & & \\ \theta(z), & \text{if } z \in V; & & \\ (z - z_0)^p 2^p \delta^{-p}, & \text{if } p \geqslant 0 & \text{and} & z \in \bar{U}_{z_0}(\delta/2); \\ (\bar{z} - \bar{z}_0)^{-p} 2^{-p} \delta^p, & \text{if } p < 0 & \text{and} & z \in \bar{U}_{z_0}(\delta/2). \end{array}$$

If $p \geqslant 0$, then $\mu(f_0, z_0) = n(z^p) = p = \mu(f, z_0)$. If $p < 0$, then $\mu(f_0, z_0) = n(\bar{z}^{-p}) = n(z^p) = p = \mu(f, z_0)$. If f vanished also at some z_1 ($\in U$, $\neq z_0$), we modify f_0 in a neighborhood of z_1, as we did above for f and z_0. We continue in this fashion, corresponding to all remaining zeros of f in U. As a result we obtain an admissible function $g \in C(\bar{U})$, having the following properties:

(1) $g(z) = f(z)$ throughout B and, consequently, $n(f) = n(g)$.

(2) If $z \in U$, then $g(z) = 0$ if and only if $f(z) = 0$.

(3) $\mu(f, z) = \mu(g, z)$ whenever $z \in U$ and $f(z) = 0$.

(4) There exists $g_0 \in C'(\bar{U})$ such that, for every $z \in \bar{U}$, $g(z) = g_0(z) \cdot \Pi_{f(x)=0,\, p=\mu(g,x) \geqslant 0}(z - x)^p \, \Pi_{f(x)=0,\, p=\mu(g,x)<0}(\bar{z} - \bar{x})^{-p}$.

Let $x \in U$. Then $y = \bar{x}^{-1} \in K - \bar{U}$, and from Theorem 3, $n((z - x)^p) = p$ and $n(z - y) = 0$. By Theorem 4, if $p < 0$, $n((\bar{z} - \bar{x})^{-p}) = -p \cdot n(\bar{z} - \bar{x}) = -p \cdot n((y - z)/(\bar{x}z)) = p$. By Theorems 4 and 6,

$$n(f) = n(g) = n(g_0) + \sum_{f(x)=0,\, p=\mu(g,x) \geqslant 0} n((z - x)^p)$$
$$+ \sum_{f(x)=0,\, p=(g,x)<0} n((\bar{z} - \bar{x})^{-p}) = \sum_{f(x)=0} \mu(f, x) = \mu(f).$$

COROLLARY (a "Rouché theorem"). *Let f, g ($\in C(\bar{U})$) be admissible, and let $|f(z) - g(z)| < |f(z)|$ throughout B. Then $\mu(f) = \mu(g)$.*

Indeed, by Theorems 13 and 4, $\mu(f) = n(f) = n(g) = \mu(g)$.

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